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The sl_2 Loop Algebra Symmetry of the Six-Vertex Model at Roots of Unity

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We demonstrate that the six vertex model (XXZ spin chain) with $\Delta = (q+q^{-1})/2$ and $q^{2N}=1$ has an invariance under the loop algebra of sl_2 which produces a special set of degenerate eigenvalues. For $\Delta = 0$ we compute the multiplicity of the degeneracies using Jordan–Wigner techniques.

KEY WORDS: 6-Vertex model; Bethe's Ansatz; quantum groups; loop algebras.

1. INTRODUCTION

The free energy of the six vertex model[(1–7)] and the eigenvalues of the related XXZ spin chain[(8–13)] specified by

$$H = \frac{1}{2} \sum_{j=1}^{L} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z) \tag{1.1}$$

have been studied for many decades by means of Bethe's Ansatz.[(8)] These results obtained from Bethe's Ansatz were shown by Baxter[(14, 15)] to follow from functional equations which closely follow from the star triangle equations and commuting transfer matrices. However, it has also been shown by Baxter[(16–18)] that if we write

$$\Delta = \frac{1}{2}(q+q^{-1}) \tag{1.2}$$

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and let

$$q^{2N} = 1 \tag{1.3}$$

then there are additional properties of the model which do not follow from the star triangle equation and commuting transfer matrices alone. Some of these additional properties have been exploited in the construction of the RSOS models[19] but, somewhat surprisingly, a complete analysis of the additional symmetries of (1.1) when the root of unity condition (1.3) holds has never been given.

We have found that when the root of unity condition (1.3) holds the Hamiltonian of the XXZ chain with periodic boundary conditions commutes with the generators of the sl_2 loop algebra and thus the space of eigenvectors decomposes into a direct sum of finite dimensional representations of loop sl_2. Moreover all of these finite dimensional representations are made up either from singlets or spin 1/2 representations and the dimensions of the degenerate subspaces are all powers of 2. This algebra is very closely related with the algebra originally used by Onsager[20] to solve the Ising model.

In this paper we will derive this loop sl_2 symmetry at roots of unity (1.3) and use the symmetry to study the degeneracies of the eigenvalues both of the transfer matrix of the six vertex model and the Hamiltonian of the XXZ spin chain. The case $N=2$ ($\Delta=0$) has also been solved long ago[21] by the technique of the Jordan–Wigner transformation. From this solution it is explicitly seen that there are many degeneracies in the spectrum which we review in Section 2. In Section 3 we present the sl_2 loop algebra which generalizes the degeneracy of Section 2 to arbitrary N. In Section 4 we treat the symmetry algebra for $N=2$ by means of the Jordan–Wigner techniques of Section 2. We conclude in Section 5 with a discussion of how the representation theory of loop sl_2 determines the degeneracy of (1.1) at roots of unity and discuss the difference between the Jordan–Wigner and the Bethe's ansatz solution for the $\Delta=0$ problem.

2. THE JORDAN–WIGNER SOLUTION OF LIEB, SCHULTZ, AND MATTIS

In 1961 Lieb, Schultz and Mattis[21] computed the eigenvalues of the XY Hamiltonian

$$H_{XY} = \frac{1}{2} \sum_{j=1}^{L} \left((1+\gamma)\, \sigma_j^x \sigma_{j+1}^x + (1-\gamma)\, \sigma_j^y \sigma_{j+1}^y \right) \tag{2.1}$$

by use of operator methods and the Jordan–Wigner transformation[22] which reduces the Hamiltonian (2.1) involving Pauli spin matrices σ_j^i ($i = x, y, z$) to a quadratic form in anticommuting fermionic operators. When $\gamma = 0$ the XY Hamiltonian reduces to (1.1) with $\Delta = 0$. In this case the total z component of the spin

$$S^z = \tfrac{1}{2} \sum_{j=1}^{L} \sigma_j^z \tag{2.2}$$

commutes with H, and in the basis where σ_j^z is diagonal (specified by the notation $|\pm 1\rangle_j$ with $\sigma_j^z \,|\pm 1\rangle_j = \pm \,|\pm 1\rangle_j$) the number of down spins n is related to S^z by

$$n = \frac{L}{2} - S^z \tag{2.3}$$

The energy eigenvalues for a given value of $S^z \geqslant 0$ are

$$E = \sum_{i=1}^{n} 2 \cos p_i \tag{2.4}$$

and the corresponding momenta are

$$P = \sum_{j=1}^{n} p_j \qquad (\mathrm{mod}\ 2\pi) \tag{2.5}$$

where the p_j obey the exclusion principle for momenta of free fermions

$$p_i \neq p_j \qquad \text{for} \quad i \neq j \tag{2.6}$$

and are freely chosen from

$$p_i \in \begin{cases} \dfrac{2\pi m}{L},\ m = 0, 1,..., L-1 & \text{for } n \text{ odd} \\[2ex] \dfrac{\pi}{L}(2m+1),\ m = 0, 1,..., L-1 & \text{for } n \text{ even} \end{cases} \tag{2.7}$$

We also note the reflection symmetry

$$E(S^z) = E(-S^z) \qquad \text{for} \quad S^z \neq 0 \tag{2.8}$$

The eigenvalue spectrum given by (2.4)–(2.7) has an important symmetry. Consider two p_j such that

$$p_1 + p_2 = \pi \qquad (\text{mod } 2\pi) \tag{2.9}$$

Then

$$\cos p_1 + \cos p_2 = 0 \tag{2.10}$$

and thus the energy of a state with given S^z is degenerate with the state with $S^z - 2$ obtained by adding the pair (2.9).

Pairs satisfying (2.9) exist for all n both even and odd and the multiplicity of the degeneracy may be computed from (2.7). For definiteness we consider $L/2$ to be an integer. Consider first S^z even. Then if $L/2$ is even (odd) p_j are obtained from the second (first) case in (2.7), the values $\pi/2$ and $3\pi/2$ do not occur and there are $L/2$ possible pairs which satisfy (2.9). Call S^z_{max} the value of S^z for which there are no pairs satisfying (2.9). This state has $L/2 - S^z_{\text{max}}$ unpaired p_j and therefore the number of possible pairs which can be formed is

$$\frac{L}{2} - \left(\frac{L}{2} - S^z_{\text{max}}\right) = S^z_{\text{max}} \tag{2.11}$$

Thus if we add l pairs which do satisfy (2.9) to the state with S^z_{max} we obtain a state with

$$S^z = S^z_{\text{max}} - 2l \qquad 0 \leqslant l \leqslant S^z_{\text{max}} \tag{2.12}$$

and a degeneracy of

$$\binom{S^z_{\text{max}}}{l} \tag{2.13}$$

If S^z is odd and $L/2$ is even (odd) then the p_j are obtained from the first (second) case in (2.7). Now the values $p = \pi/2$ and $3\pi/2$ can occur. These special values give zero contribution to the energy but cannot participate in a pair of the form (2.9). Thus in this case there are $L/2 - 1$ possible pairs. Consider first the case where the state S^z_{max} contains either $p = \pi/2$ or $3\pi/2$ but not both. Then there are

$$\frac{L}{2} - 1 - \left(\frac{L}{2} - S^z_{\text{max}} - 1\right) = S^z_{\text{max}} \tag{2.14}$$

possible pairs and thus if l pairs are added to the state we see that (2.12) and (2.13) continue to hold.

Secondly consider the case where $S^z_{\max}$ does not contain either $p=\pi/2$ or $3\pi/2$. Then there are only

$$\frac{L}{2}-1-\left(\frac{L}{2}-S^z_{\max}\right)=S^z_{\max}-1 \tag{2.15}$$

number of possible pairs and thus if we add l pairs we obtain a state with

$$S^z=S^z_{\max}-2l \qquad 0\leqslant l\leqslant S^z_{\max}-1 \tag{2.16}$$

and a degeneracy of

$$\binom{S^z_{\max}-1}{l} \tag{2.17}$$

Finally let $S^z_{\max}$ contain both $\pi/2$ and $3\pi/2$. Then the number of possible pairs is

$$\frac{L}{2}-1-\left(\frac{L}{2}-S^z_{\max}-2\right)=S^z_{\max}+1 \tag{2.18}$$

and if we add l pairs to this state we obtain the state

$$S^z=S^z_{\max}-2l \qquad 0\leqslant l\leqslant S^z_{\max}+1 \tag{2.19}$$

and a degeneracy of

$$\binom{S^z_{\max}+1}{l} \tag{2.20}$$

These last two cases are equivalent by use of the reflection symmetry (2.8).

3. THE SYMMETRY OF THE LOOP ALGEBRA sl_2 FOR GENERAL N

Degeneracies of eigenvalues are caused by symmetries of the system. The XY model solved by LSM[21] is equivalent to a free Fermi problem and this is a very powerful symmetry which leads not only to the degeneracies of Section 2 but to many other degeneracies as well.

In an XXZ chain (1.1) of finite length L the free particle degeneracies are destroyed by letting $\Delta\neq 0$. However the degeneracies of Section 2 are special in that they exist in systems of finite length when the root of unity condition (1.3) holds for q. We have studied these degeneracies for chains up to length $L=16$ for $N=3$, 4 and 5 and found that there are sets of degenerate eigenvalues for values of S^z satisfying the generalization of (2.12) to arbitrary N

$$S^z = S^z_{\mathrm{max}} - Nl \tag{3.1}$$

If

$$S^z \equiv 0 (\mathrm{mod}\ N) \tag{3.2}$$

then

$$0 \leqslant l \leqslant 2S^z_{\mathrm{max}}/N \tag{3.3}$$

and the degeneracy is

$$\binom{2S^z_{\mathrm{max}}/N}{l} \tag{3.4}$$

which reduces to (2.13) when $N=2$. If

$$S^z \not\equiv 0 (\mathrm{mod}\ N) \tag{3.5}$$

then just as in Section 2 there are three cases

$$0 \leqslant l \leqslant 2[S^z_{\mathrm{max}}/N] + (0, 1, 2) \tag{3.6}$$

which can occur and the degeneracies are

$$\binom{2[S^z_{\mathrm{max}}/N] + (0, 1, 2)}{l} \tag{3.7}$$

respectively where $[x]$ is the greatest integer contained in x.

These degeneracies must also be produced by a symmetry of the model and, of course, the XXZ chain is known to follow from the commuting transfer matrix symmetry algebra of the six vertex model. But this holds for all values of q whereas the degeneracies we are here discussing exist only when q satisfies the root of unity condition (1.3). Thus we conclude that there must be a further symmetry beyond the commutation relations of the transfer matrix at generic q.

We have found that this symmetry algebra is the loop algebra of sl_2. In this section we will define the algebra and show how it is related to the XXZ and six vertex model.

The study of the symmetries of the XXZ chain with periodic boundary conditions at roots of unity (1.3) was initiated in ref. 23 where it was seen that even though the Hamiltonian (1.1) is not invariant under the full quantum group $U_q(sl_2)$ it is invariant under a suitable smaller set of operators. To be specific we recall the spin 1/2 representation of the generators of $U_q(sl_2)$ (following ref. 24)

$$q^{S^z} = q^{\sigma^z/2} \otimes \cdots \otimes q^{\sigma^z/2} \tag{3.8}$$

$$S^{\pm} = \sum_{j=1}^{L} S_j^{\pm} = \sum_{j=1}^{L} q^{\sigma^z/2} \otimes \cdots q^{\sigma^z/2} \otimes \sigma_j^{\pm} \otimes q^{-\sigma^z/2} \otimes \cdots \otimes q^{-\sigma^z/2} \tag{3.9}$$

which satisfy the relations of $U_q(sl_2)$

$$q^{S^z} S^{\pm} q^{-S^z} = q^{\pm 1} S^{\pm} \tag{3.10}$$

$$[S^+, S^-] = \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}} \tag{3.11}$$

and by definition $(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2$.

The nth power of the operators $S^{\pm}$ satisfy

$$\begin{aligned} S^{\pm n} &= q^{n(n-1)/2}[n]! \sum_{1 \leqslant j_1 < j_2 < \cdots < j_n \leqslant L} S_{j_1}^{\pm} S_{j_2}^{\pm} \cdots S_{j_n}^{\pm} \\ &= [n]! \sum_{1 \leqslant j_1 < \cdots < j_n \leqslant L} q^{(n/2)\, \sigma^z} \otimes \cdots \otimes q^{(n/2)\, \sigma^z} \\ &\quad \otimes \sigma_{j_1}^{\pm} \otimes q^{((n-2)/2)\, \sigma^z} \otimes \cdots \otimes q^{((n-2)/2)\, \sigma^z} \\ &\quad \otimes \sigma_{j_2}^{\pm} \otimes q^{((n-4)/2)\, \sigma^z} \otimes \cdots \otimes \sigma_{j_n}^{\pm} \otimes q^{-(n/2)\, \sigma^z} \otimes \cdots \otimes q^{-(n/2)\, \sigma^z} \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} &[n] = (q^n - q^{-n})/(q - q^{-1}) \qquad \text{for} \quad n > 0 \quad \text{and} \quad [0] = 1 \\ &[n]! = \prod_{k=1}^{n} [k] \end{aligned} \tag{3.13}$$

and we have used

$$q^{A\sigma^z} \sigma^{\pm} = q^{\pm A} \sigma^{\pm} \tag{3.14}$$

Then when the root of unity condition (1.3) holds we have

$$(S^{\pm})^N = 0 \tag{3.15}$$

However if we set[25–27]

$$S^{\pm(N)} = \lim_{q^{2N}\to 1} (S^{\pm})^N/[N]! \tag{3.16}$$

then the operators $S^{\pm(N)}$ exist and are non-vanishing and in particular

$$\begin{aligned} S^{\pm(N)} = \sum_{1\leqslant j_1<\cdots<j_N\leqslant L} & q^{(N/2)\,\sigma^z}\otimes\cdots\otimes q^{(N/2)\,\sigma^z} \\ & \otimes\sigma_{j_1}^{\pm}\otimes q^{((N-2)/2)\,\sigma^z}\otimes\cdots\otimes q^{((N-2)/2)\,\sigma^z} \\ & \otimes\sigma_{j_2}^{\pm}\otimes q^{((N-4)/2)\,\sigma^z}\otimes\cdots\otimes\sigma_{j_N}^{\pm}\otimes q^{-(N/2)\,\sigma^z} \\ & \otimes\cdots\otimes q^{-(N/2)\,\sigma^z} \end{aligned} \tag{3.17}$$

It is shown in ref. 23 that for S^z/N an integer that $S^{\pm(N)}$ commutes with the Hamiltonian (1.1) when (1.3) holds.

We have found that $S^{\pm(N)}$ is not just an isolated operator which commutes with the Hamiltonian (1.1) and transfer matrix of the six vertex model but in fact is part of a much larger symmetry algebra. The key to this symmetry algebra is the observation of Jimbo[24] that the representation of $S^{\pm}$ given by (3.9) is not unique but that there exists an equally good isomorphic representation

$$T^{\pm} = \sum_{j=1}^{L} T_j^{\pm} = \sum_{j=1}^{L} q^{-\sigma^z/2}\otimes\cdots q^{-\sigma^z/2}\otimes\sigma_j^{\pm}\otimes q^{\sigma^z/2}\otimes\cdots\otimes q^{\sigma^z/2} \tag{3.18}$$

which is obtained from $S^{\pm}$ by the replacement $q\to q^{-1}$. In the case considered here where q is a root of unity (1.3) $T^{\pm}$ and $S^{\pm}$ are related by complex and hermitian conjugation

$$T^{\pm} = S^{\pm *} = S^{\mp\dagger} \tag{3.19}$$

which may be thought of as a dual transformation and by

$$T^{\pm} = RS^{\mp}R^{-1} \tag{3.20}$$

where $R=\sigma_1^x\otimes\sigma_2^x\otimes\cdots\otimes\sigma_L^x$ is the spin inversion operator. The operator $T^{\pm}$ also has the property that when the root of unity condition (1.3) holds then $(T^{\pm})^N=0$ and we define $T^{\pm(N)}$ exactly as we defined $S^{\pm(N)}$ by (3.16) and (3.17) as

$$\begin{aligned} T^{\pm(N)} = \sum_{1\leqslant j_1<\cdots<j_N\leqslant L} & q^{-(N/2)\sigma^z}\otimes\cdots\otimes q^{-(N/2)\sigma^z}\otimes\sigma^{\pm}_{j_1} \\ & \otimes q^{-((N-2)/2)\sigma^z}\otimes\cdots\otimes q^{-((N-2)/2)\sigma^z}\otimes\sigma^{\pm}_{j_2} \\ & \otimes q^{-((N-4)/2)\sigma^z}\otimes\cdots\otimes\sigma^{\pm}_{j_N}\otimes q^{(N/2)\sigma^z}\otimes\cdots\otimes q^{(N/2)\sigma^z} \end{aligned} \tag{3.21}$$

By use of the theory of quantum groups it follows from the work of Jimbo[24] that the elementary commutation relations between $S^{\pm}$ and $T^{\pm}$ hold for all q

$$[S^+, T^+]=[S^-, T^-]=0 \tag{3.22}$$

as well as the four quantum Serre relations

$$\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right] S^{+3}T^- - \left[\begin{smallmatrix}3\\1\end{smallmatrix}\right] S^{+2}T^-S^+ + \left[\begin{smallmatrix}3\\2\end{smallmatrix}\right] S^+T^-S^{+2} - \left[\begin{smallmatrix}3\\3\end{smallmatrix}\right] T^-S^{+3}=0 \tag{3.23}$$

$$\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right] S^{-3}T^+ - \left[\begin{smallmatrix}3\\1\end{smallmatrix}\right] S^{-2}T^+S^- + \left[\begin{smallmatrix}3\\2\end{smallmatrix}\right] S^-T^+S^{-2} - \left[\begin{smallmatrix}3\\3\end{smallmatrix}\right] T^+S^{-3}=0 \tag{3.24}$$

$$\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right] T^{+3}S^- - \left[\begin{smallmatrix}3\\1\end{smallmatrix}\right] T^{+2}S^-T^+ + \left[\begin{smallmatrix}3\\2\end{smallmatrix}\right] T^+S^-T^{+2} - \left[\begin{smallmatrix}3\\3\end{smallmatrix}\right] S^-T^{+3}=0 \tag{3.25}$$

$$\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right] T^{-3}S^+ - \left[\begin{smallmatrix}3\\1\end{smallmatrix}\right] T^{-2}S^+T^- + \left[\begin{smallmatrix}3\\2\end{smallmatrix}\right] T^-S^+T^{-2} - \left[\begin{smallmatrix}3\\3\end{smallmatrix}\right] S^+T^{-3}=0 \tag{3.26}$$

(3.27)

where we define

$$\begin{aligned} \begin{bmatrix} m \\ l \end{bmatrix} &= \frac{[m]!}{[l]!\,[m-l]!} = \prod_{k=1}^{l} \frac{q^{m-l+k}-q^{-(m-l+k)}}{q^k-q^{-k}} \qquad \text{for} \quad 0<l<m \\ &= 1 \qquad \text{for} \quad l=0, m \end{aligned} \tag{3.28}$$

We will give elementary proofs of these fundamental commutation relations which do not explicitly rely on quantum group theory later in this section.

From these equations we specialize to $q^{2N}=1$ to derive

$$[S^{+(N)}, T^{+(N)}]=[S^{-(N)}, T^{-(N)}]=0 \tag{3.29}$$

$$[S^{\pm(N)}, S^z]=\pm NS^{\pm(N)}, \qquad [T^{\pm(N)}, S^z]=\pm NT^{\pm(N)} \tag{3.30}$$

and, by use of the higher order Serre relations of Lusztig,[28] we will show that

$$S^{+(N)3}T^{-(N)} - 3S^{+(N)2}T^{-(N)}S^{+(N)} + 3S^{+(N)}T^{-(N)}S^{+(N)2} - T^{-(N)}S^{+(N)3} = 0 \tag{3.31}$$

$$S^{-(N)3}T^{+(N)} - 3S^{-(N)2}T^{+(N)}S^{-(N)} + 3S^{-(N)}T^{+(N)}S^{-(N)2} - T^{+(N)}S^{-(N)3} = 0 \tag{3.32}$$

$$T^{+(N)3}S^{-(N)} - 3T^{+(N)2}S^{-(N)}T^{+(N)} + 3T^{+(N)}S^{-(N)}T^{+(N)2} - S^{-(N)}T^{+(N)3} = 0 \tag{3.33}$$

$$T^{-(N)3}S^{+(N)} - 3T^{-(N)2}S^{+(N)}T^{-(N)} + 3T^{-(N)}S^{+(N)}T^{-(N)2} - S^{+(N)}T^{-(N)3} = 0 \tag{3.34}$$

In the sector $S^z \equiv 0 (\mathrm{mod}\ N)$ we additionally, by use of results from ref. 29, show that

$$[S^{+(N)}, S^{-(N)}] = [T^{+(N)}, T^{-(N)}] = -(-q)^N \frac{2}{N} S^z \tag{3.35}$$

Proofs will be given later in this section.

If we make the identifications

$$\begin{aligned} e_0 &= S^{+(N)}, \qquad f_0 = S^{-(N)}, \qquad e_1 = T^{-(N)} \\ f_1 &= T^{+(N)}, \qquad t_0 = -t_1 = -(-q)^N S^z/N \end{aligned} \tag{3.36}$$

and if we do not impose the relation (3.19) and the identity $t_0 = -t_1$ which was demanded by (3.35) we see that equations (3.29)–(3.35) are the defining relations[29] of the Chevalley generators of the affine Lie algebra $A_1^{(1)'}$. The identity $t_0 = -t_1$ reduces this algebra to the defining relations of the Chevalley generators of the loop algebra of sl_2.

The theory of sl_2 loop algebra and of its finite dimensional representations may now be applied to the XXZ model when q satisfies the root of unity condition (1.3) by noting the commutation relation with the Hamiltonian (1.1) which holds when $S^z \equiv 0 (\mathrm{mod}\ N)$

$$[S^{\pm(N)}, H] = [T^{\pm(N)}, H] = 0 \tag{3.37}$$

More generally we have translational (anti)invariance of the operators $S^{\pm(N)}$, $T^{\pm(N)}$

$$S^{\pm(N)}e^{iP} = q^N e^{iP} S^{\pm(N)}, \qquad T^{\pm(N)}e^{iP} = q^N e^{iP} T^{\pm(N)} \tag{3.38}$$

where the momentum operator P is defined from the shift operator

$$\Pi_L|_{\{j\},\{j'\}} = \prod_{i=1}^{L} \delta_{j_i, j'_{i-1}} \tag{3.39}$$

as $\Pi_L = e^{-iP}$ and we will prove in Appendix A the (anti) commutation relation

$$S^{\pm(N)}T(v) = q^N T(v)\, S^{\pm(N)}, \qquad T^{\pm(N)}T(v) = q^N T(v)\, T^{\pm(N)} \tag{3.40}$$

where $T(v)$ is the six vertex model transfer matrix. Thus

$$[S^{\pm(N)}, T(v)\, e^{-iP}] = [T^{\pm(N)}, T(v)\, e^{-iP}] = 0 \tag{3.41}$$

which reduces to (3.37) as $e^v \to 1$. Therefore the spectrum of the transfer matrix of the six vertex model and the spectrum of the XXZ model decomposes into finite dimensional representations of the sl_2 loop algebra which are given explicitly on p. 243 of ref. 29.

The discussion of the previous section indicates that we should expect that for sectors where $S^z \neq 0 (\mathrm{mod}\ N)$ the existence of an sl_2 loop algebra is somewhat more involved. We begin by noting that in the sector $S^z \equiv n (\mathrm{mod}\ N)$ where $1 \leqslant n \leqslant N-1$ the following four operators are translationally invariant and commute with the transfer matrix

$$(T^+)^n (S^-)^n, \quad (S^+)^n (T^-)^n, \quad (T^-)^{N-n} (S^+)^{N-n}, \quad (S^-)^{N-n} (T^+)^{N-n} \tag{3.42}$$

Furthermore even though $S^{\pm(N)}$ and $T^{\pm(N)}$ do not (anti) commute with the transfer matrix that the eight operators

$$\begin{aligned}
&(T^+)^n\, (S^-)^n\, S^{-(N)}, && S^{-(N)} (S^-)^{N-n}\, (T^+)^{N-n} \\
&S^{+(N)} (S^+)^n\, (T^-)^n, && (T^-)^{N-n}\, (S^+)^{N-n}\, S^{+(N)} \\
&T^{+(N)} (T^+)^n\, (S^-)^n, && (S^-)^{N-n}\, (T^+)^{N-n}\, T^{+(N)} \\
&(S^+)^n\, (T^-)^n\, T^{-(N)}, && T^{-(N)} (T^-)^{N-n}\, (S^+)^{N-n}
\end{aligned} \tag{3.43}$$

do (anti) commute both with e^{iP} and the transfer matrix in the sector $S^z \equiv n(\mathrm{mod}\ N)$.

The operators of (3.42) each have a large null space but they are not themselves projection operators. However on the computer (at least for $N=3$) we have numerically constructed the projection operators onto the eigenspace of the nonzero eigenvalues of the operators (3.42) and using (3.43) have constructed the corresponding projections of $S^{\pm(N)}$ and $T^{\pm(N)}$ and have verified that for the projected operators in the sector $S^z \equiv n(\mathrm{mod}\ N)$ with $1 \leqslant n \leqslant N-1$ we find that (3.29)–(3.34) hold without modification but (3.35) is slightly modified.

We also note that the case of $N=2$ is special in that the operator T^+S^- satisfies

$$(T^+S^-)^2 = LT^+S^- \tag{3.44}$$

for all values of S^z (and not just $S^z \equiv 1(\mathrm{mod}\ 2)$) and similarly for the other three operators S^+T^-, T^-S^+, S^-T^+. Thus (up to a factor of L) these operators for $N=2$ are already projection operators.

We conclude this section with the proofs of (3.22)–(3.35) and (3.38).

3.1. Proof of (3.22) and (3.29)

To prove (3.22) we use the definitions (3.9) and (3.18) and the fact that $(\sigma^+)^2=0$ to write

$$\begin{aligned}
[S^+, T^+] = \sum_{j_1<j_2} \{ & (q^{\sigma^z/2}\sigma^+_{j_1}) \otimes q^{\sigma^z} \otimes \cdots \otimes q^{\sigma^z} \otimes (\sigma^+_{j_2} q^{\sigma^z/2}) \\
& + (\sigma^+_{j_1} q^{-\sigma^z/2}) \otimes q^{-\sigma^z} \otimes \cdots \otimes q^{-\sigma^z} \otimes (q^{-\sigma^z/2}\sigma^+_{j_2}) \\
& - (q^{-\sigma^z/2}\sigma^+_{j_1}) \otimes q^{-\sigma^z} \otimes \cdots \otimes q^{-\sigma^z} \otimes (\sigma^+_{j_2} q^{-\sigma^z/2}) \\
& - (\sigma^+_{j_1} q^{\sigma^z/2}) \otimes q^{\sigma^z} \otimes \cdots \otimes q^{\sigma^z} \otimes (q^{\sigma^z/2}\sigma^+_{j_2})\}
\end{aligned} \tag{3.45}$$

If we now use the commutation relation (3.10) it is seen that terms 1 and 4 and terms 2 and 3 cancel in this sum. Thus (3.22) follows. Equation (3.29) follows immediately from (3.22).

3.2. Proof of (3.23)–(3.26)

To give an elementary proof of (3.23) we divide by $[3]!$ and use both (3.12) and the companion equation for $T^{\pm n}$ to write

$$
\begin{aligned}
&(S^{+3}/[3]!)\,T^- - (S^{+2}/[2]!)\,T^-S^+ + S^+T^-(S^{+2}/[2]!) - T^-(S^{+3}/[3]!)\\
&\quad = \left(\sum_{1\leqslant j_1<j_2<j_3\leqslant L} q^{3/2\sigma^z}\otimes\sigma_{j_1}^+\otimes q^{1/2\sigma^z}\otimes\sigma_{j_2}^+\otimes q^{-1/2\sigma^z}\otimes\sigma_{j_3}^+\otimes q^{-3/2\sigma^z}\right)\\
&\qquad \times\left(\sum_{1\leqslant j_4\leqslant L} q^{-1/2\sigma^z}\otimes\sigma_{j_4}^-\otimes q^{1/2\sigma^z}\right)\\
&\qquad -\left(\sum_{1\leqslant j_1<j_2\leqslant L} q^{\sigma^z}\otimes\sigma_{j_1}^+\otimes I\otimes\sigma_{j_2}^+\otimes q^{-\sigma^z}\right)\\
&\qquad \times\left(\sum_{1\leqslant j_4\leqslant L} q^{-1/2\sigma^z}\otimes\sigma_{j_4}^-\otimes q^{1/2\sigma^z}\right)\left(\sum_{1\leqslant j_3\leqslant L} q^{1/2\sigma^z}\otimes\sigma_{j_3}^+\otimes q^{-1/2\sigma^z}\right)\\
&\qquad +\left(\sum_{1\leqslant j_1\leqslant L} q^{1/2\sigma^z}\otimes\sigma_{j_1}^+\otimes q^{-1/2\sigma^z}\right)\left(\sum_{1\leqslant j_4\leqslant L} q^{-1/2\sigma^z}\otimes\sigma_{j_4}^-\otimes q^{1/2\sigma^z}\right)\\
&\qquad \times\left(\sum_{1\leqslant j_2<j_3\leqslant L} q^{\sigma^z}\otimes\sigma_{j_2}^+\otimes I\otimes\sigma_{j_3}^+\otimes q^{-\sigma^z}\right)\\
&\qquad -\left(\sum_{1\leqslant j_4\leqslant L} q^{-1/2\sigma^z}\otimes\sigma_{j_4}^-\otimes q^{1/2\sigma^z}\right)\\
&\qquad \times\left(\sum_{1\leqslant j_1<j_2<j_3\leqslant L} q^{3/2\sigma^z}\otimes\sigma_{j_1}^+\otimes q^{1/2\sigma^z}\otimes\sigma_{j_2}^+\otimes q^{-1/2\sigma^z}\otimes\sigma_{j_3}^+\otimes q^{-3/2\sigma^z}\right)
\end{aligned}
\tag{3.46}
$$

where we have used $q^{a\sigma^z}$ to denote $q^{a\sigma^z}\otimes\cdots\otimes q^{a\sigma^z}$.

We now note that in the expansion of this expression there are two types of terms, those where $\sigma_{j_4}^-$ never is at the same site as one of the σ_j^+ and those where $\sigma_{j_4}^-$ and at least one of the σ_j^+ are at the same site. These two types are treated separately.

For the first type of term there are four distinct cases depending on the location of the $\sigma_{j_4}^-$ relative to the three σ_j^+. For example consider the term where $\sigma_{j_4}^-$ lies to the right of the three σ_j^+. Then each of the four terms in (3.46) gives a contribution of the form

$$
A_j \sum_{1\leqslant j_1<j_2<j_3<j_4\leqslant L} q^{\sigma^z}\sigma_{j_1}^+\otimes I\otimes\sigma_{j_2}^+\otimes q^{-\sigma^z}\otimes\sigma_{j_3}^+\otimes q^{-2\sigma^z}\otimes\sigma_{j_4}^-\otimes q^{-\sigma^z}
\tag{3.47}
$$

where by use of (3.14) it is elementary to find

$$
A_1=q^3,\qquad A_2=-q^3-q-q^{-1},\qquad A_3=q+q^{-1}+q^{-3},\qquad A_4=-q^{-3}
\tag{3.48}
$$

We see that $A_1+A_2+A_3+A_4=0$ and thus the contribution from these terms vanished.

Similar elementary computations demonstrate similar cancellations in all other cases and thus (3.23) is demonstrated. The other Serre relations (3.24)–(3.26) follow in the identical manner.

3.3. Proof of (3.35)

We begin the proof of (3.35) by noting the commutation relation valid for general q (1.3.1) on p. 474 of ref. 27 for the case of the group $U_q(sl_2)$

$$[(S^+)^m, (S^-)^n] = \sum_{j=1}^{\min(m,n)} \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} [j]!\, (S^-)^{n-j} (S^+)^{m-j} \times \prod_{k=0}^{j-1} \frac{q^{2S^z+m-n-k} - q^{-(2S^z+m-n-k)}}{q-q^{-1}} \tag{3.49}$$

Thus setting $m=n=N$ and dividing by $[N]!^2$ we obtain

$$[(S^+)^N/[N]!, (S^-)^N/[N]!] = \sum_{j=1}^{N} \frac{1}{[N-j]!^2\,[j]!} (S^-)^{N-j} (S^+)^{N-j} \times \prod_{k=0}^{j-1} \frac{q^{2S^z-k} - q^{-2S^z+k}}{q-q^{-1}} \tag{3.50}$$

In the sector $S^z \equiv 0 \bmod N$ the $k=0$ term in the product above vanishes in the limit when $q^{2N} \to 1$. Thus the only terms in the sum over j which can fail to vanish are those where the coefficient of the product diverges. This occurs only for $j=N$. Thus using the definition (3.16) we may let $q^{2N} \to 1$ to find

$$\begin{aligned}
[S^{+(N)}, S^{-(N)}] &= \lim_{q^{2N}\to 1} \frac{1}{[N]!} \prod_{k=0}^{N-1} \frac{q^{2S^z-k} - q^{-2S^z+k}}{q-q^{-1}} \\
&= (-1)^{N-1} \lim_{q^{2N}\to 1} \frac{q^{2S^z} - q^{-2S^z}}{q^N - q^{-N}} \\
&= -(-q)^N \frac{2}{N} S^z
\end{aligned} \tag{3.51}$$

as desired.

3.4. Proof of (3.31)–(3.34)

To proof (3.31)–(3.34) we begin with a result of Lusztig ((7.1.6) on p. 57 of ref. 28) on higher order q-Serre relations. That result when specialized to the quantum group $U_q(\widehat{sl_2})$ says that for the operators S^+ and T^- which satisfy the quantum Serre relation (3.23) it follows from properties of the algebra alone (and not of the representation) that for any q if we set

$$\begin{aligned} \theta_1^{(m)} &= (S^+)^m/[m]! \\ \theta_2^{(m)} &= (T^-)^m/[m]! \end{aligned} \tag{3.52}$$

then we have

$$\theta_1^{(3N)}\theta_2^{(N)} = \sum_{s'=N}^{3N} \gamma_{s'}\theta_1^{(3N-s')}\theta_2^{(N)}\theta_1^{(s')} \tag{3.53}$$

where

$$\gamma_{s'} = (-1)^{s'+1} q^{s'(N-1)} \sum_{l=0}^{N-1} (-1)^l q^{l(1-s')} \begin{bmatrix} s' \\ l \end{bmatrix} \tag{3.54}$$

We obtain the Serre relation (3.31) taking the limit $q^{2N} \to 1$ in (3.53). Thus we write

$$s' = Ns + p \qquad \text{with} \quad p = 0, 1, ..., N-1 \tag{3.55}$$

and by use of (3.28) we see that

$$\lim_{q^{2N}\to 1} \begin{bmatrix} sN+p \\ l \end{bmatrix} = \begin{cases} q^{Nsl} \begin{bmatrix} p \\ l \end{bmatrix} & \text{for} \quad p \geqslant l \\ 0 & \text{otherwise} \end{cases} \tag{3.56}$$

Therefore we obtain

$$\lim_{q^{2N}\to 1} \gamma_{Ns+p} = (-1)^{Ns+p+1} q^{(Ns+p)(N-1)} \sum_{l=0}^{p} (-1)^l q^{l(1-p)} \begin{bmatrix} p \\ l \end{bmatrix} \tag{3.57}$$

However from the q-binomial theorem (for example 1.34 of ref. 28) we have

$$\sum_{l=0}^{p} (-1)^l q^{l(1-p)} \begin{bmatrix} p \\ l \end{bmatrix} = \delta_{p,0} \tag{3.58}$$

for all q and thus we obtain

$$\lim_{q^{2N} \to 1} \gamma_{Ns+p} = \delta_{p,0}(-1)^{Ns+1} q^{Ns(N-1)} = \delta_{p,0}(-1)^{s+1} \tag{3.59}$$

where in the last line we have used $q^N = \pm 1$ for N odd and $q^N = -1$ for N even.

If we now note from (3.52) that

$$\theta^{(Ns)} = \frac{[N]!^s}{[Ns]!} \theta^{(N)\,s} \tag{3.60}$$

and use the relation derived from (3.13) that

$$\lim_{q^{2N} \to 1} \frac{[N]!^s}{[Ns]!} = \frac{q^{Ns(s-1)/2}}{s!} \tag{3.61}$$

we may use (3.59) in (3.53) and restore the definition (3.52) to find

$$\sum_{s=0}^{3} \frac{(-1)^{s+1}}{s!\,(3-s)!} S^{+(N)\,3-s} T^{-(N)} S^{+(N)\,s} = 0 \tag{3.62}$$

from which (3.31) follows immediately. The proof of the remaining Serre relations (3.32)–(3.34) is identical.

3.5. Proof of Translational (Anti)Invariance of $S^{\pm(N)}$ in $S^z \equiv 0 \pmod N$ (3.38) and $(T^+)^n (S^-)^n$ in $S^z \equiv n \pmod N$ (3.42)

Let us denote by Π_R the inverse of the shift operator Π_L. By definition of the shift operator (3.39) we have for any set of operators A_j in the jth position in the tensor product

$$\Pi_R A_1 \otimes A_2 \otimes \cdots \otimes A_L \Pi_R^{-1} = A_2 \otimes \cdots \otimes A_L \otimes A_1 \tag{3.63}$$

Considering the actions of the shift operator Π_R on $S_j^{\pm}$ $(T_j^{\pm})$ for $j = 1,..., L$, explicitly, we have

$$\Pi_R S^{\pm} \Pi_R^{-1} = (S^{\pm} - S_L^{\pm})\, q^{\sigma_L^z} + S_L^{\pm} q^{-2S^z + \sigma_L^z} \tag{3.64}$$

$$\Pi_R T^{\pm} \Pi_R^{-1} = (T^{\pm} - T_L^{\pm})\, q^{-\sigma_L^z} + T_L^{\pm} q^{2S^z - \sigma_L^z} \tag{3.65}$$

and we note that σ^z_L commutes with $(S^\pm - S^\pm_L)$ and $(T^\pm - T^\pm_L)$. Taking the nth powers of (3.64) and (3.65), we find, for all q

$$(\Pi_R S^\pm \Pi_R^{-1})^n$$
$$= \{(S^\pm)^n + q^{\pm(N-1)}[n](S^\pm)^{n-1}\, S^\pm_L(q^{-2S^z} - 1)\}\, q^{n\sigma^z_L} \tag{3.66}$$
$$= q^{n\sigma^z_L}\{(S^\pm)^n + q^{\pm(n-1)}[n](S^\pm)^{n-1}\, S^\pm_L(q^{-2(S^z \pm n)} - 1)\} \tag{3.67}$$
$$(\Pi_R T^\pm \Pi_R^{-1})^n$$
$$= \{(T^\pm)^n + q^{\mp(n-1)}[n](T^\pm)^{n-1}\, T^\pm_L(q^{2S^z} - 1)\}\, q^{-n\sigma^z_L} \tag{3.68}$$
$$= q^{-n\sigma^z_L}\{(T^\pm)^n + q^{\mp(n-1)}[n](T^\pm)^{n-1}\, T^\pm_L(q^{2(S^z \pm n)} - 1)\} \tag{3.69}$$

The commutation relations (3.38) for $S^z \equiv 0 (\mathrm{mod}\ N)$ now follow from (3.67) and (3.69) by letting $n = N$, dividing by $[N]!$, taking the limit $q^{2N} \to 1$ and using $q^{-2(S^z \pm N)} = 1$.

Similarly for $(S^+)^n (T^-)^n$ we find from (3.66)–(3.69) for arbitrary q

$$\begin{aligned} &\Pi_R (S^+)^n (T^-)^n \\ &\quad = (\Pi_R S^+ \Pi_R^{-1})^n (\Pi_R T^- \Pi_R^{-1})^n\, \Pi_R \\ &\quad = \{(S^+)^n + q^{n-1}[n](S^+)^{n-1}\, S^+_L(q^{-2S^z} - 1)\}\, q^{n\sigma^z_L} \\ &\qquad \times q^{-n\sigma^z_L}\{(T^-)^n + q^{-(n-1)}[n](T^-)^{n-1}\, T^-_L(q^{2(S^z - n)} - 1)\}\, \Pi_R \end{aligned} \tag{3.70}$$

Thus if we use $q^{S^z}(T^-)^n = (T^-)^n\, q^{S^z - n}$ we find for $S^z = n > 0$ that for all q

$$[\Pi_R, (S^+)^n (T^-)^n] = 0 \tag{3.71}$$

When the root of unity condition $q^{2N} = 1$ holds this argument immediately extends to $S^z = n (\mathrm{mod}\ N)$.

4. A JORDAN–WIGNER PROOF OF THE SYMMETRY FOR $N = 2$

In this section we will prove all of the loop sl_2 commutation relations for $N = 2$ by use of the Jordan–Wigner operators used by LSM[21] in the computation of the eigenvalue spectrum discussed in Section 2. This construction provides insight into the general representation theory of the loop algebra sl_2 and into the projection operators needed for $S^z \equiv 1 (\mathrm{mod}\ 2)$.

4.1. Notation

We begin by recalling the Jordan–Wigner transformation of ref. 21. The Hamiltonian (1.1) with $\Delta=0$

$$H=\tfrac{1}{2}\sum_{j=1}^{L}(\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y) \tag{4.1}$$

is first written in terms of $\sigma_j^{\pm}=\frac{1}{2}(\sigma_j^x\pm i\sigma_j^y)$ as

$$H=\sum_{j=1}^{L}(\sigma_j^+\sigma_{j+1}^-+\sigma_j^-\sigma_{j+1}^+) \tag{4.2}$$

We then define the Jordan–Wigner transformation to operators c_j and $c_j^\dagger$

$$\begin{aligned}
c_j&=\exp\left(i\pi\sum_{k=1}^{j-1}\sigma_k^+\sigma_k^-\right)\sigma_j^-=e^{(\pi i/2)(j-1)}\exp\left(\frac{i\pi}{2}\sum_{k=1}^{j-1}\sigma_k^z\right)\sigma_j^-\\
&=e^{(-\pi i/2)(j-1)}\exp\left(\frac{-i\pi}{2}\sum_{k=1}^{j-1}\sigma_k^z\right)\sigma_j^-\\
c_j^\dagger&=\exp\left(-i\pi\sum_{k=1}^{j-1}\sigma_k^+\sigma_k^-\right)\sigma_j^+\\
&=e^{-(\pi i/2)(j-1)}\exp\left(\frac{-i\pi}{2}\sum_{k=1}^{j-1}\sigma_k^z\right)\sigma_j^+=e^{(\pi i/2)(j-1)}\exp\left(\frac{i\pi}{2}\sum_{k=1}^{j-1}\sigma_k^z\right)\sigma_j^+
\end{aligned} \tag{4.3}$$

with

$$c_j^\dagger c_j=\sigma_j^+\sigma_j^-=\tfrac{1}{2}(1+\sigma_j^z) \tag{4.4}$$

which has the inverse

$$\sigma_j^-=\exp\left(i\pi\sum_{k=1}^{j-1}c_k^\dagger c_k\right)c_j,\qquad \sigma_j^+=\exp\left(-i\pi\sum_{k=1}^{j-1}c_k^\dagger c_k\right)c_j^\dagger \tag{4.5}$$

Here we have used the fact that because the eigenvalues of $\sigma_k^+\sigma_k^-$ are only zero and one, the reality condition holds

$$c_j=c_j^* \tag{4.6}$$

Furthermore the c_j and $c_j^\dagger$ satisfy Fermi canonical anticommuation relations

$$\{c_j, c_{j'}^\dagger\} = \delta_{j,j'} \qquad \text{and} \qquad \{c_j, c_{j'}\} = \{c_j^\dagger, c_{j'}^\dagger\} = 0 \tag{4.7}$$

In terms of these new operators the Hamiltonian (4.1) becomes

$$H = \sum_{i=1}^{L-1} (c_i^\dagger c_{i+1} - c_i c_{i+1}^\dagger) + (c_L^\dagger c_1 - c_L c_1^\dagger)\, e^{\pi i (S^z + L/2 - 1)} \tag{4.8}$$

where the final term may be interpreted as a periodic or antiperiodic boundary condition by defining

$$c_{L+1} = c_1 e^{\pi i (l+1)} \qquad \text{with} \quad l = S^z + L/2 \tag{4.9}$$

where l is the number of up spins.

The two different boundary conditions on (4.8) are treated together by introducing for $l \equiv 0, 1 \pmod 2$ the Fourier transform operators $\eta_p^{(l)}$ and $\eta_p^{(l)\dagger}$ by

$$\eta_p^{(l)} = \frac{1}{\sqrt{L}} \sum_{k=1}^{L} \exp(-ikp)\, c_k, \qquad \eta_p^{(l)\dagger} = \frac{1}{\sqrt{L}} \sum_{k=1}^{L} \exp(ikp)\, c_k^\dagger \tag{4.10}$$

with the inverse

$$c_k = \frac{1}{\sqrt{L}} \sum_p \exp(ikp)\, \eta_p^{(l)}, \qquad c_k^\dagger = \frac{1}{\sqrt{L}} \sum_p \exp(-ikp)\, \eta_p^{(l)\dagger} \tag{4.11}$$

where for p we may use either of the allowed sets of p which follow from the periodicity requirements $c_{k+L} = (-1)^{(l+1)} \exp(iLp)\, c_k$

$$p = \begin{cases} \dfrac{\pi}{L}(2m+1),\ 0 \leqslant m \leqslant L-1 & \text{for} \quad l = 0 \\[2ex] \dfrac{\pi}{L} 2m,\ 0 \leqslant m \leqslant L-1 & \text{for} \quad l = 1 \end{cases} \tag{4.12}$$

just as in Section 2. From the reality condition (4.6) we have

$$\eta_p^* = \eta_{2\pi - p}, \qquad \eta_p^{*\dagger} = \eta_{2\pi - p}^\dagger \tag{4.13}$$

from (4.7) and (4.10) we obtain

$$\{\eta_p^{(l)}, \eta_{p'}^{(l)\dagger}\} = \delta_{p,p'} \qquad \text{and} \qquad \{\eta_p^{(l)}, \eta_{p'}^{(l)}\} = \{\eta_{p'}^{(l)\dagger}, \eta_p^{(l)\dagger}\} = 0 \tag{4.14}$$

and from (4.8) we find

$$H_l = 2 \sum_p \cos(p)\, \eta_p^{(l)\dagger} \eta_p^{(l)} \tag{4.15}$$

The eigenstates of H_l are

$$\eta_{p_1}^{(l)\dagger} \eta_{p_2}^{(l)\dagger} \cdots \eta_{p_n}^{(l)\dagger} \,|0\rangle \tag{4.16}$$

where $|0\rangle$ is the state with all L spins down and the number of fermions

$$n = \sum_{m=1}^{L} c_m^\dagger c_m = \sum_p \eta_p^\dagger \eta_p = L/2 + S^z \tag{4.17}$$

takes the values 0, 1,..., L. The number of states (4.16) is 2^L. To obtain the coordinate representations of H-eigenstates we use (4.10) and (4.3).

4.2. The Operators of the Loop Algebra sl_2

For $N=2$ the operators $S^\pm$ and $T^\pm$ are defined from (3.9) and (3.18) and the operators $S^{\pm(2)}$ and $T^{\pm(2)}$ are defined from (3.17) and (3.21) with $q = e^{\pi i/2}$. We may write them explicitly in terms of c_j and $c_j^\dagger$ as

$$\begin{aligned} S^+ &= e^{\pi i/4} \left(\sum_{j=1}^{L} e^{-\pi i j/2} c_j^\dagger \right) e^{-\pi i S^z/2} = e^{3\pi i/4} e^{-\pi i S^z/2} \sum_{j=1}^{L} e^{-\pi i j/2} c_j^\dagger \\ T^+ &= e^{-\pi i/4} \left(\sum_{j=1}^{L} e^{\pi i j/2} c_k^\dagger \right) e^{\pi i S^z/2} = e^{-3\pi i/4} e^{\pi i S^z/2} \sum_{j=1}^{L} e^{\pi i j/2} c_j^\dagger \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} S^{+(2)} &= \sum_{1 \leqslant j < k \leqslant L} e^{(\pi i/2) \sum_{m=1}^{j-1} \sigma_m^z} \sigma_j^+ \sigma_k^+ e^{-(\pi i/2) \sum_{m=k+1}^{L} \sigma_m^z} \\ &= i e^{\pi i S^z} \sum_{1 \leqslant j < k \leqslant L} e^{-(\pi i/2)(j+k)} c_j^\dagger c_k^\dagger \end{aligned} \tag{4.19}$$

$$\begin{aligned} T^{+(2)} &= \sum_{1 \leqslant j < k \leqslant L} e^{-(\pi i/2) \sum_{m=1}^{j-1} \sigma_m^z} \sigma_j^+ \sigma_k^+ e^{(\pi i/2) \sum_{m=k+1}^{L} \sigma_m^z} \\ &= -i e^{\pi i S^z} \sum_{1 \leqslant j < k \leqslant L} e^{(\pi i/2)(j+k)} c_j^\dagger c_k^\dagger \end{aligned} \tag{4.20}$$

and S^-, T^-, $S^{-(2)}$ and $T^{-(2)}$ are obtained by the replacements $\sigma_j^+ \to \sigma_j^-$ and $c_j^\dagger \to c_j$ in (4.18), (4.19) and (4.20). Here we have used the identity $\sigma_j^z/2 = \sigma_j^+\sigma_j^- - 1/2$, and the definition (4.3).

We may now use the Fourier transformed operators (4.10) and for the operators $S^\pm$, $T^\pm$ we easily find

$$\begin{aligned} S^+ &= L^{1/2} e^{\pi i/4} \eta^\dagger_{3\pi/2} e^{-\pi i S^z/2} = L^{1/2} e^{3\pi i/4} e^{-\pi i S^z/2} \eta^\dagger_{3\pi/2} \\ T^+ &= L^{1/2} e^{-\pi i/4} \eta^\dagger_{\pi/2} e^{\pi i S^z/2} = L^{1/2} e^{-3\pi i/4} e^{\pi i S^z/2} \eta^\dagger_{\pi/2} \\ S^- &= L^{1/2} e^{3\pi i/4} \eta_{\pi/2} e^{-\pi i S^z/2} = L^{1/2} e^{\pi i/4} e^{-\pi i S^z/2} \eta_{\pi/2} \\ T^- &= L^{1/2} e^{-3\pi i/4} \eta_{3\pi/2} e^{\pi i S^z/2} = L^{1/2} e^{-\pi i/4} e^{\pi i S^z/2} \eta_{3\pi/2} \end{aligned} \tag{4.21}$$

From this equation and the anticommutation relations (4.14) we find that

$$[S^+, T^+] = -iL\{\eta^\dagger_{3\pi/2}, \eta^\dagger_{\pi/2}\} = 0, \qquad [S^-, T^-] = -iL\{\eta_{\pi/2}, \eta_{3\pi/2}\} = 0 \tag{4.22}$$

which is the commutation relation (3.22). In addition we see from (4.21) that

$$\begin{aligned} S^+T^- &= L\eta^\dagger_{3\pi/2}\eta_{3\pi/2}, \qquad & T^-S^+ &= L\eta_{3\pi/2}\eta^\dagger_{3\pi/2} \\ T^+S^- &= L\eta^\dagger_{\pi/2}\eta_{\pi/2}, \qquad & S^-T^+ &= L\eta_{\pi/2}\eta^\dagger_{\pi/2} \end{aligned} \tag{4.23}$$

from which we get the projection operator relations (3.44)

$$\begin{aligned} (S^+T^-)^2 &= LS^+T^-, \qquad & (T^-S^+)^2 &= LT^-S^+ \\ (T^+S^-)^2 &= LT^+S^-, \qquad & (S^-T^+)^2 &= LS^-T^+ \end{aligned} \tag{4.24}$$

For the operators $S^{\pm(2)}$, $T^{\pm(2)}$ we first write

$$\begin{aligned} S^{+(2)} &= \sum_{p_1, p_2} A^{(l)}(p_1, p_2)\, \eta^{(l)\dagger}_{p_1} \eta^{(l)\dagger}_{p_2} \\ T^{+(2)} &= \sum_{p_1, p_2} A^{(l)*}(-p_1, -p_2)\, \eta^{(l)\dagger}_{p_1} \eta^{(l)\dagger}_{p_2} \\ S^{-(2)} &= -\sum_{p_1, p_2} A^{(l)}(-p_1, -p_2)\, \eta^{(l)}_{p_1} \eta^{(l)}_{p_2} \\ T^{-(2)} &= -\sum_{p_1, p_2} A^{(l)*}(p_1, p_2)\, \eta^{(l)}_{p_1} \eta^{(l)}_{p_2} \end{aligned} \tag{4.25}$$

where

$$A^{(l)}(p_1, p_2) = \frac{ie^{-\pi i S^z}}{2L} \sum_{1 \leqslant j < k \leqslant L} \{e^{-ij(\pi/2 + p_1)} e^{-ik(\pi/2 + p_2)} - e^{-ij(\pi/2 + p_2)} e^{-ik(\pi/2 + p_1)}\} \tag{4.26}$$

We have antisymmetrized this expression because of the anticommutation relations (4.14). However to proceed further we need to know if for some p we can have

$$p + \frac{\pi}{2} \equiv 0 (\text{mod } 2\pi) \tag{4.27}$$

The possibility of this holding depends both on l and on whether or not $L/2$ is even or odd.

Consider first the case where there is no possible value of p for which (4.27) can hold. We see from (4.12) that this occurs either if $l = 0$ and $L/2$ is even or $l = 1$ and $L/2$ is odd. We see from (4.15) that in this case the operators $\eta_p^{(l)}$ are the operators in H_l for $S^z \equiv 0 (\text{mod } 2)$. Then we do the sum over j in (4.26) and find

$$A^{(l)} = \frac{ie^{-\pi i S^z}}{2L} \sum_{1 \leqslant k \leqslant L} \left\{ e^{-ik(\pi/2 + p_2)} \frac{e^{-i(\pi/2 + p_1)} - e^{-ik(\pi/2 + p_1)}}{1 - e^{-i(\pi/2 + p_1)}} - e^{-ik(\pi/2 + p_1)} \frac{e^{-i(\pi/2 + p_2)} - e^{-k(\pi/2 + p_2)}}{1 - e^{-i(\pi/2 + p_2)}} \right\} \tag{4.28}$$

The sum on k is now also a geometric series whose value depends on whether or not

$$\pi + p_1 + p_2 \equiv 0 (\text{mod } 2\pi) \tag{4.29}$$

which is allowed in both cases $l = 0$ with $L/2$ even and $l = 1$ with $L/2$ odd even though (4.27) can not hold. If there are no p which satisfy (4.29) it is easy to see that since $e^{iL(\pi/2 + p)} = -1$ holds in both cases that the sum over k vanishes. However if (4.29) does hold the sum does not vanish and hence we find

$$A^{(l)}(p_1, p_2) = -\frac{e^{-i\pi S^z}}{2} \cot \frac{1}{2}\left(p_1 + \frac{\pi}{2}\right) \delta_{p_1 + p_2 + \pi, 0} \tag{4.30}$$

Thus we explicitly find from (4.25)

$$\begin{aligned}
S^{+(2)} &= -\frac{e^{-i\pi S^z}}{2} \sum_p \cot \frac{1}{2}\left(p+\frac{\pi}{2}\right) \eta_p^{(l)\dagger} \eta_{\pi-p}^{(l)\dagger} \\
T^{+(2)} &= -\frac{e^{i\pi S^z}}{2} \sum_p \tan \frac{1}{2}\left(p+\frac{\pi}{2}\right) \eta_p^{(l)\dagger} \eta_{\pi-p}^{(l)\dagger} \\
S^{-(2)} &= \frac{e^{-i\pi S^z}}{2} \sum_p \tan \frac{1}{2}\left(p+\frac{\pi}{2}\right) \eta_p^{(l)} \eta_{\pi-p}^{(l)} \\
T^{-(2)} &= \frac{e^{i\pi S^z}}{2} \sum_p \cot \frac{1}{2}\left(p+\frac{\pi}{2}\right) \eta_p^{(l)} \eta_{\pi-p}^{(l)}
\end{aligned} \tag{4.31}$$

where we note that $[\eta_p^{(l)} \eta_{\pi-p}^{(l)}, e^{i\pi S^z}] = 0$.

The degeneracy of the spectrum of H in the sector $S^z \equiv 0 (\mathrm{mod}\ 2)$ where $p \neq \pi/2,\ 3\pi/2$ presented in Section 2 is now very transparently demonstrated. The key to this demonstration is to consider the operators

$$\eta_p^{(l)} \eta_{\pi-p}^{(l)}, \qquad \eta_p^{(l)\dagger} \eta_{\pi-p}^{(l)\dagger} \tag{4.32}$$

which appear in the summands of the representations of the operators $S^{\pm(2)}$, $T^{\pm(2)}$. Using the representation of the Hamiltonian (4.15) and the commutation relations (4.14) we find

$$\begin{aligned}
[\eta_p^{(l)\dagger} \eta_{\pi-p}^{(l)\dagger}, H_l] &= 2 \cos p [\eta_p^{(l)\dagger} \eta_{\pi-p}^{(l)\dagger}, \eta_p^{(l)\dagger} \eta_p^{(l)}] \\
&\quad + 2 \cos(\pi - p) [\eta_p^{(l)\dagger} \eta_{\pi-p}^{(l)\dagger}, \eta_{\pi-p}^{(l)\dagger} \eta_{\pi-p}^{(l)}] \\
&= -2 \cos p \eta_p^{(l)\dagger} \eta_{\pi-p}^{(l)\dagger} - 2 \cos(\pi - p)\, \eta_p^{(l)\dagger} \eta_{\pi-p}^{(l)\dagger} = 0
\end{aligned} \tag{4.33}$$

Thus the pair operator $\eta_p^{(l)\dagger} \eta_{\pi-p}^{(l)\dagger}$ adds an excitation of zero energy with $S^z = 2$ (and momentum π) to the system. Furthermore

$$(\eta_p^{(l)\dagger} \eta_{\pi-p}^{(l)\dagger})^2 = 0 \tag{4.34}$$

so that only one pair is allowed for each value of p. The properties (4.33) and (4.34) reproduce the degeneracy found in Section 2.

It is now a simple matter to use (4.14) and (4.31) to compute the commutation relations needed for the loop algebra of sl_2 given in (3.29), (3.31)–(3.35). We first obtain the single commutators

$$[S^{+(2)}, T^{+(2)}] = [S^{-(2)}, T^{-(2)}] = 0 \tag{4.35}$$

$$[S^{+(2)}, S^{-(2)}] = [T^{-(2)}, T^{-(2)}] = \sum_p \left(\eta_p^{(l)\dagger}\eta_p^{(l)} - \frac{1}{2}\right) = S^z \tag{4.36}$$

$$[S^{+(2)}, T^{-(2)}] = \sum_p \cot^2 \frac{1}{2}\left(p + \frac{\pi}{2}\right)\left(\eta_p^{(l)\dagger}\eta_p^{(l)} - \frac{1}{2}\right) \tag{4.37}$$

$$[T^{+(2)}, S^{-(2)}] = \sum_p \tan^2 \frac{1}{2}\left(p + \frac{\pi}{2}\right)\left(\eta_p^{(l)\dagger}\eta_p^{(l)} - \frac{1}{2}\right) \tag{4.38}$$

then the double commutators

$$[T^{-(2)}, [T^{-(2)}, S^{+(2)}]] = -e^{\pi i S^z} \sum_p \cot^3 \frac{1}{2}\left(p + \frac{\pi}{2}\right) \eta_p^{(l)}\eta_{\pi-p}^{(l)} \tag{4.39}$$

$$[T^{+(2)}, [T^{+(2)}, S^{-(2)}]] = e^{\pi i S^z} \sum_p \tan^3 \frac{1}{2}\left(p + \frac{\pi}{2}\right) \eta_p^{(l)\dagger}\eta_{\pi-p}^{(l)\dagger} \tag{4.40}$$

$$[T^{+(2)}, [T^{-(2)}, S^{+(2)}]] = 2S^{+(2)} \tag{4.41}$$

$$[T^{-(2)}, [T^{+(2)}, S^{-(2)}]] = 2S^{-(2)} \tag{4.42}$$

and finally the triple commutators

$$\begin{aligned} &[T^{-(2)}, [T^{-(2)}, [T^{-(2)}, S^{+(2)}]]] \\ &\quad = [T^{+(2)}, [T^{+(2)}, [T^{+(2)}, S^{-(2)}]]] = 0 \end{aligned} \tag{4.43}$$

$$\begin{aligned} &[S^{-(2)}, [S^{-(2)}, [S^{-(2)}, T^{+(2)}]]] \\ &\quad = [S^{+(2)}, [S^{+(2)}, [S^{+(2)}, T^{-(2)}]]] = 0 \end{aligned} \tag{4.44}$$

$$\begin{aligned} &[T^{+(2)}, [T^{-(2)}, [T^{-(2)}, S^{+(2)}]]] \\ &\quad = [T^{-(2)}, [T^{+(2)}, [T^{-(2)}, S^{+(2)}]]] = 2[T^{-(2)}, S^{+(2)}] \end{aligned} \tag{4.45}$$

$$\begin{aligned} &[T^{+(2)}, [T^{-(2)}, [T^{+(2)}, S^{-(2)}]]] \\ &\quad = [T^{-(2)}, [T^{+(2)}, [T^{+(2)}, S^{-(2)}]]] = 2[T^{+(2)}, S^{-(2)}] \end{aligned} \tag{4.46}$$

where the single commutators(4.35) and (4.36) are (3.29) and (3.35). The triple commutators (4.43), (4.44) are the Serre relations (3.31)–(3.34) of the algebra and follow immediately from (4.31), (4.39) and (4.40) by use of (4.34).

On the other hand if $l = 0$ with $L/2$ odd or $l = 1$ with $L/2$ even then the condition (4.27) can hold. In both cases we see from (4.15) that the operators $\eta_p^{(l)}$ are those of H_l with $S^z \equiv 1 (\mathrm{mod}\ 2)$. There are now two cases to consider depending on whether or not (4.27) is satisfied.

If there are no values of p which satisfy (4.27) then by using $e^{iL(\pi/2 + p)} = 1$ we find as before that $A^{(l)}(p_1, p_2)$ is given by (4.30). But if either p_1 or p_2 is $3\pi/2$ and $p_1+p_2+\pi \neq 0 (\mathrm{mod}\ 2\pi)$ we find that $A^{(l)}(p_1, p_2)$ does not vanish and that instead we have

$$A^{(l)}\left(\frac{3\pi}{2}, p\right) = -A^{(l)}\left(p, \frac{3\pi}{2}\right) = -\frac{ie^{-\pi i S^z}}{1-e^{-i(\pi/2+p)}} \tag{4.47}$$

Thus instead of (4.31) we have

$$\begin{aligned} S^{+(2)} = \frac{-e^{-i\pi S^z}}{2} \sum_{p\neq 3\pi/2} \Bigg(& \eta_p^{(l)\dagger}\eta_{\pi-p}^{(l)\dagger} \cot\frac{1}{2}\left(p+\frac{\pi}{2}\right) \\ & -2\eta_{3\pi/2}^{(l)\dagger}\eta_p^{(l)\dagger}\frac{i}{1-e^{-i(\pi/2+p)}}\Bigg) \end{aligned} \tag{4.48}$$

The second term in (4.48) does not commute with the Hamiltonian and as discussed in Section 2 we need instead to use the projection operators (4.24) to construct the projected operators

$$\begin{aligned} S_{pr}^{+(2)} &= S^{+(2)}S^+T^- + T^-S^+S^{+(2)} \\ S_{pr}^{-(2)} &= S^{-(2)}S^-T^+ + T^+S^-S^{-(2)} \\ T_{pr}^{+(2)} &= T^{+(2)}T^+S^- + S^-T^+T^{+(2)} \\ T_{pr}^{-(2)} &= T^{+(2)}T^-S^+ + S^+T^-T^{-(2)} \end{aligned} \tag{4.49}$$

These projected operators are readily expressed in terms of $\eta_p^{(l)}$ by use of (4.48) and the identity

$$\eta_p\eta_p^\dagger\eta_{p_1}^\dagger\eta_{p_2}^\dagger + \eta_{p_1}^\dagger\eta_{p_2}^\dagger\eta_p^\dagger\eta_p = \eta_{p_1}^\dagger\eta_{p_2}^\dagger(1-\delta_{p,p_1}-\delta_{p,p_2}) \tag{4.50}$$

Thus we find that the noncommuting terms of $\eta_{3\pi/2}^\dagger\eta_p^\dagger$ and $\eta_{\pi/2}^\dagger\eta_p^\dagger$ are annihilated by the projection operator and we obtain the expressions analogous to the unprojected expressions (4.31)

$$\begin{aligned} S_{pr}^{+(2)} &= -\frac{e^{-i\pi S^z}}{2}\sum_{p\neq\pi/2,\,3\pi/2} \cot\frac{1}{2}\left(p+\frac{\pi}{2}\right)\eta_p^{(l)\dagger}\eta_{\pi-p}^{(l)\dagger} \\ T_{pr}^{+(2)} &= -\frac{e^{i\pi S^z}}{2}\sum_{p\neq\pi/2,\,3\pi/2} \tan\frac{1}{2}\left(p+\frac{\pi}{2}\right)\eta_p^{(l)\dagger}\eta_{\pi-p}^{(l)\dagger} \\ S_{pr}^{-(2)} &= -\frac{e^{-i\pi S^z}}{2}\sum_{p\neq\pi/2,\,3\pi/2} \tan\frac{1}{2}\left(p+\frac{\pi}{2}\right)\eta_p^{(l)}\eta_{\pi-p}^{(l)} \\ T_{pr}^{-(2)} &= -\frac{e^{i\pi S^z}}{2}\sum_{p\neq\pi/2,\,3\pi/2} \cot\frac{1}{2}\left(p+\frac{\pi}{2}\right)\eta_p^{(l)}\eta_{\pi-p}^{(l)} \end{aligned} \tag{4.51}$$

These operators manifestly commute with the Hamiltonian and the computation of the degeneracy is identical with the previous case of $S^z \equiv 0 (\bmod 2)$. Similarly all the commutation relations of the sl_2 loop algebra (4.35)–(4.46) hold for the projected operators with the one exception of (4.36) where we find the slight modification

$$
\begin{aligned}
[S^{+(2)}, S^{-(2)}] = [T^{+(2)}, T^{-(2)}] &= \sum_{p \neq \pi/2,\, 3\pi/2} (\eta_p^{(l)\dagger} \eta_p^{(l)} - \tfrac{1}{2}) \\
&= S^z - (\eta_{\pi/2}^{(l)\dagger} \eta_{\pi/2}^{(l)} - \tfrac{1}{2}) - (\eta_{3\pi/2}^{(l)\dagger} \eta_{3\pi/2}^{(l)} - \tfrac{1}{2}) \qquad (4.52)
\end{aligned}
$$

5. DISCUSSION

In Section 4 we have demonstrated that the operators $S^{\pm(2)}$ and $T^{\pm(2)}$ obtained from the quantum group $U_q(\widehat{sl_2})$ at $q = e^{i\pi/2}$ both explain the degeneracies of the spectrum of the Hamiltonian found in Section 2 and obey the defining commutation relations of the loop algebra sl_2 given in Section 3. The reason for this more elaborate treatment of the degeneracies computed by elementary means in Section 2 is that for $N > 2$ where the treatment of Section 2 no longer is possible we have found that the sl_2 symmetry algebra of Section 2 still holds and therefore the space of eigenstates of the Hamiltonian (1.1) decomposes into a direct sum of evaluation representations of the loop algebra of sl_2. This decomposition is explicitly contained in the representation given in (4.31) for the unprojected operators and (4.51) for their projected counterparts. In this expression the fact that $(\eta_p^\dagger \eta_{\pi-p}^\dagger)^2 = 0$ is equivalent to the statement that only spin 1/2 representations occur.

The demonstration that only spin 1/2 representations occur is more complicated than the demonstration of the sl_2 loop algebra symmetry. For $N = 2$ we can in principle compute sufficiently many multiple commutators and show that they are of the form of the right hand side of (4.40) with $\tan^3(p + \pi/2)/2$ replaced by various other powers. Thus we can generate L equations for the L different operators $\eta_p^\dagger \eta_{\pi-p}^\dagger$ and solve the system. In practice what we did in Section 4 was to examine the summands which appeared in the expressions for $S^{\pm(2)}$ and $T^{\pm(2)}$ and then discovered that these summands also commuted with the Hamiltonian. The ability to do this for $N \geqslant 3$ relies on having a proper form for $S^{\pm(N)}$.

In this paper the proper form relied on the Jordan–Wigner operators and thus it seems profitable to generalize the fermionic operators c_j to "parafermionic" operators in position space

$$\begin{aligned}
c_j &= q^{2\sum_{k=1}^{j-1}\sigma_k^+\sigma_k^-}\sigma_j^- = q^{j-1}q^{\sum_{k=1}^{j-1}\sigma_k^z}\sigma_j^- \\
c_j^* &= q^{-2\sum_{k=1}^{j-1}\sigma_k^+\sigma_k^-}\sigma_j^- = q^{-(j-1)}q^{-\sum_{k=1}^{j-1}\sigma_k^z}\sigma_j^- \\
c_j^\dagger &= q^{-2\sum_{k=1}^{j-1}\sigma_k^+\sigma_k^-}\sigma_j^+ = q^{-(j-1)}q^{-\sum_{k=1}^{j-1}\sigma_k^z}\sigma_j^+ \\
c_j^{*\dagger} &= q^{2\sum_{k=1}^{j-1}\sigma_k^+\sigma_k^-}\sigma_j^+ = q^{j-1}q^{\sum_{k=1}^{j-1}\sigma_k^z}\sigma_j^+
\end{aligned} \tag{5.1}$$

This generalization can be carried out along the lines of the treatment of parafermions in ref. 30 but gives somewhat cumbersome expressions for the (anti) commutators of the generalization of the η_p.

A second possibility is to consider instead of the Jordan–Wigner operators the Bethe's ansatz wave function as given (for example) by Yang and Yang.[13] This wave function is of the form

$$\psi = \sum_P A_P e^{i\sum_j k_{Pj}x_j} \tag{5.2}$$

where the sum is over all permutations P and

$$A_p(2\Delta e^{-ip_1} - 1 - e^{-(p_1+p_2)}) = -A_{p'}(2\Delta e^{-ip_2} - 1 - e^{-(p_1+p_2)}) \tag{5.3}$$

for permutations P and P' which differ only in the interchange of the two adjacent elements p_1 and p_2. If we naively set $\Delta = 0$ we find $A_P = -A_{P'}$ and thus ψ is a Slater determinant wave function.

Now the coordinate space form of the Jordan–Wigner wave function is also a Slater determinant and thus it might be expected that the Bethe's wavefunctions at $\Delta = 0$ and the Jordan–Wigner wavefunction are identical. It is most important to realize, however, that this is in fact not the case when we are considering degenerate eigenvalues. This can be seen very explicitly by considering the operation of the spin reflection operator R on the states with $S^z = 0$. All the Bethe's wave functions are eigenstates of R for $\Delta \neq 0$ and thus continue to be eigenfunctions at $\Delta = 0$. But a direct computation shows that the Jordan–Wigner states are not eigenstates of R. Moreover if the solutions of the Bethe's equations of ref. 13 for $\Delta \neq 0$ are smoothly continued to $\Delta = 0$ we have explicitly found for the degenerate eigenvalues that there are complex solutions to Bethe's equations which remain complex even in the limit $\Delta = 0$.

Part of what is happening is that in the degenerate subspace there are solutions with pairs p_1 and p_2 where $p_1 + p_2 = \pi$. But when this condition is put into (5.3) and then we set $\Delta = 0$ we see that (5.3) reduces to $0 = 0$ and the relation between A_P and $A_{P'}$ is no longer determined. Indeed it is exactly this loophole in the argument of ref. 13 which is exploited by

Baxter[16–18] in his computation of the eigenvectors of the XYZ model at roots of unity.

We thus conclude that for degenerate eigenstates the Jordan–Wigner states are linear combinations of the Bethe's states and that the mechanism needed to recognize the spin 1/2 nature of the representations will depend on which set of basis states is used.

We note further that a numerical study of the eigenvalues of the XYZ spin chain indicates that when the root of unity condition $q^{2N}=1$ is generalized to Baxter's condition ((C15) of ref. 15) of

$$2N\eta = 2m_1 K + im_2 K' \tag{5.4}$$

that the size of the degenerate multiplets of the XXZ model are diminished by at most a factor of two. This clearly indicates that with an appropriate generalization of the operators $S^{\pm(N)}$ and $T^{\pm(N)}$ the sl_2 loop symmetry of the XXZ model extends to the XYZ model.

In addition we have found that all commutation relations with the transfer matrix proven in this paper continue to hold in the more general case where the L spectral variables of the vertical lines of the 2^L dimensional row transfer matrix are allowed to be arbitrary instead of being equal.

APPENDIX A. COMMUTATION (ANTI-COMMUTATION) RELATIONS WITH THE TRANSFER MATRIX

In this appendix we show that the transfer matrix $T(v)$ of the six-vertex model can be written as a sum of products of the Temperley–Lieb generators multiplied by the shift operators and furthermore that for all q the operators $S^{\pm}$ and $T^{\pm}$ commute with the Temperley–Lieb operators. Thus, any operator constructed from $S^{\pm}$ and $T^{\pm}$ commutes (anti-commutes) with the transfer matrix if it (anti-)commutes with the shift operators. Therefore since we proved in Section 3.5 that $S^{\pm(N)}$ and $T^{\pm(N)}$ (anti)-commutes with the shift operator and that the operators given by (3.42) and (3.43) (anti)-commute with the shift operator the (anti-)commutation with the full transfer matrix follows.

A.1. Twisted Transfer Matrix of the Six-Vertex Model

There are many different sets of local Boltzmann weights which give the same transfer matrix. These Boltzmann weights differ by a gauge transformation. In this appendix we find it convenient to use two different gauge equivalent sets of Boltzmann weights[31] which we denote by $W^{+}(\mu, v)|_{\alpha, \beta}$ and $W^{-}(\mu, v)|_{\alpha, \beta}$ whose nonzero elements are given by

$$W^{\pm}(1, 1)|_{1,1} = W^{\pm}(-1, -1)|_{-1,-1} = -2\sinh v \tag{A1}$$

$$W^{\pm}(-1, -1)|_{1,1} = W^{\pm}(1, 1)|_{-,1-1} = 2\sinh(\lambda - v) \tag{A2}$$

$$W^{\pm}(-1, 1)|_{1,-1} = 2e^{\pm v}\sinh\lambda \tag{A3}$$

$$W^{\pm}(1, -1)|_{-1,1} = 2e^{\mp v}\sinh\lambda \tag{A4}$$

where $q = e^{\lambda}$ and we recall that $\Delta = (q + q^{-1})/2$. More generally we will also need the "twisted" Boltzmann weights with the twisting parameter ϕ by

$$\widetilde{W}^{\pm}(\mu, v; \phi)|_{\alpha,\beta} = q^{\phi(\mu+\alpha)} W^{\pm}(\mu, v)|_{\alpha,\beta} \qquad \text{for} \quad \mu, v, \alpha, \beta = \pm 1 \tag{A5}$$

The matrix elements of the "twisted" transfer matrix $T(v; \phi)$ are then defined by

$$\begin{aligned}(T(v;\phi))^{\mu_1,\dots,\mu_L}_{v_1,\dots,v_L} &= \operatorname{Tr} \widetilde{W}^{\pm}(\mu_1, v_1; \phi)\, W^{\pm}(\mu_2, v_2) \cdots W^{\pm}(\mu_L, v_L) \\ &= \sum_{\alpha_1 \cdots \alpha_L} \widetilde{W}^{\pm}(\mu_1, v_1; \phi)|_{\alpha_1,\alpha_2} W^{\pm}(\mu_2, v_2)|_{\alpha_2,\alpha_3} \cdots W^{\pm}(\mu_L, v_L)|_{\alpha_L,\alpha_1}\end{aligned} \tag{A6}$$

Due to the "*ice rule*," we have the same number of configurations for the two weights $W^{\pm}(-1, 1)|_{1,-1}$ and $W^{\pm}(1, -1)|_{-1,1}$ in the product (A6) and thus the changes in the Boltzmann weights are canceled out for the twisted transfer matrix $T(v; \phi)$.

We denote the twisted transfer matrix at $\phi = 0$ by $T(v) = T(v; \phi = 0)$ and note that the transfer matrix $T(v)$ is related to the shift operator Π_L (3.39) and the XXZ Hamiltonian (1.1) by

$$T(v \approx 0) = (q - q^{-1})^L\, \Pi_L \left\{ I - \frac{2v}{q - q^{-1}} \left(H + \frac{L(q + q^{-1})}{4} \right) + o(v) \right\} \tag{A7}$$

We note in passing that just as the sign of the Hamiltonian (1.1) may be negated by a similarity transformation that same transformation changes the sign of the Boltzmann weight (A1) and sends $T(v) \rightarrow (-1)^{L/2 - S^z}\, UT(v)\, U^{-1}$ where $U = \sigma_1^z \otimes I_2 \otimes \cdots \otimes \sigma_{L-1}^z \otimes I_L$.

A.2. Operators $X_j^{\pm}(v)$ and a Decomposition of the Transfer Matrix

We define $\tilde{X}_j^{\pm}(v;\phi)$ for $j=1,\ldots, L-1$, by the following

$$\tilde{X}_j^{\pm}(v;\phi)=\sum_{a,b,c,d=\pm 1}\tilde{X}_{bd}^{\pm ac}(v;\phi)\, I_1\otimes\cdots\otimes I_{j-1}\otimes E_j^{ab}\otimes E_{j+1}^{cd}\otimes I_{j+2}\otimes\cdots\otimes I_L \tag{A8}$$

and define

$$X_j^{\pm}(v)=\tilde{X}_j^{\pm}(v;0) \tag{A9}$$

where E^{ab} denotes the matrix

$$(E^{ab})_{c,d}=\delta_{a,c}\,\delta_{b,d}\qquad \text{for}\quad c,d=\pm 1 \tag{A10}$$

and

$$\begin{aligned}\tilde{X}_{bd}^{\pm ac}(v;\phi)&=q^{\phi(a-b)}X_{bd}^{\pm ac}(v)=q^{\phi(a-b)}W^{\pm}(a,d)\,|_{-b,-c}\\&=2\sinh(\lambda-v)\,\delta_{a,b}\,\delta_{c,d}\\&\quad+2\sinh vabe^{\mp(a+b)\lambda/2}q^{\phi(a-b)}\,\delta_{a,-c}\,\delta_{b,-d}\end{aligned} \tag{A11}$$

where we note the symmetry[32]

$$X_{b,d}^{\pm a,c}(v)=-ade^{\mp(a-d)\lambda/2}X_{-a,b}^{\pm c,-d}(\lambda-v)=-bce^{\mp(b-c)\lambda/2}X_{d,-c}^{\pm -b,a}(\lambda-v) \tag{A12}$$

Thus we find that the expression (A6) for the transfer matrix becomes

$$\begin{aligned}&(T(v;\phi))_{\nu_1\nu_2\cdots\nu_L}^{\mu_1\mu_2\cdots\mu_L}\\&\quad=\sum_{\alpha_1,\ldots,\alpha_L}\tilde{W}^{\pm}(\mu_1,\nu_1;\phi)\,|_{\alpha_1,\alpha_2}\,W^{\pm}(\mu_2,\nu_2)\,|_{\alpha_2,\alpha_3}\cdots W^{\pm}(\mu_L,\nu_L)\,|_{\alpha_L,\alpha_1}\\&\quad=\sum_{\alpha_1,\ldots,\alpha_L}\tilde{X}_{-\alpha_1,\nu_1}^{\pm\mu_1,-\alpha_2}(v;\phi)\,X_{-\alpha_2,\nu_2}^{\pm\mu_2,-\alpha_3}(v)\cdots X_{-\alpha_L,\nu_L}^{\pm\mu_L,-\alpha_1}(v)\\&\quad=\sum_{\beta_0,\ldots,\beta_{L-1}}X_{\beta_{L-1},\nu_L}^{\pm\mu_L,\beta_0}(v)\cdots X_{\beta_1,\nu_2}^{\pm\mu_2,\beta_2}(v)\,\tilde{X}_{\beta_0,\nu_1}^{\pm\mu_1,\beta_1}(v;\phi)\\&\quad=\sum_{\beta_0,\beta_1}(X_{L-1}^{\pm}(v)\cdots X_1^{\pm}(v))_{\beta_1\nu_2\cdots\nu_{L-1}\nu_L}^{\mu_2\mu_3\cdots\mu_L\beta_0}\,\tilde{X}_{\beta_0\nu_1}^{\pm\mu_1\beta_1}(v;\phi)\end{aligned} \tag{A13}$$

Substituting (A11) into the first factor $\widetilde{X}^{\pm \mu_1, \beta_1}_{\beta_0 \nu_1}$ of the last line of (A13), we have

$$
\begin{aligned}
&(T(v; \phi))^{\mu_1 \cdots \mu_L}_{\nu_1 \cdots \nu_L} \\
&\quad = 2 \sinh(\lambda - v) \sum_{\beta_0, \beta_1} \delta_{\mu_1, \beta_0} \delta_{\beta_1, \nu_1} (X^{\pm}_{L-1}(v) \, X^{\pm}_{L-2}(v) \cdots X^{\pm}_{1}(v))^{\mu_2 \cdots \mu_L \beta_0}_{\beta_1 \nu_2 \cdots \nu_L} \\
&\qquad + 2 \sinh v \sum_{\beta_0, \beta_1} \mu_1 \beta_0 e^{\mp (\mu_1 + \beta_0) \lambda/2} q^{\phi(\mu_1 - \beta_0)} \delta_{\mu_1, -\beta_1} \delta_{\beta_0, -\nu_1} \\
&\qquad \times (X^{\pm}_{L-1}(v) \, X^{\pm}_{L-2}(v) \cdots X^{\pm}_{1}(v))^{\mu_2 \cdots \mu_L \beta_0}_{\beta_1 \nu_2 \cdots \nu_L} \\
&\quad = 2 \sinh(\lambda - v)(X^{\pm}_{L-1}(v) \, X^{\pm}_{L-2}(v) \cdots X^{\pm}_{1}(v))^{\mu_2 \cdots \mu_L \mu_1}_{\nu_1 \nu_2 \cdots \nu_L} \\
&\qquad - 2 \sinh v \mu_1 \nu_1 e^{\mp (\mu_1 - \nu_1) \lambda/2} q^{\phi(\mu_1 + \nu_1)} (X^{\pm}_{L-1}(v) \\
&\qquad \times X^{\pm}_{L-2}(v) \cdots X^{\pm}_{1}(v))^{\mu_2 \cdots \mu_L, -\nu_1}_{-\mu_1 \nu_2 \cdots \nu_L} \\
&\quad = 2 \sinh(\lambda - v)(\Pi_L X^{\pm}_{L-1}(v) \, X^{\pm}_{L-2}(v) \cdots X^{\pm}_{1}(v))^{\mu_1 \mu_2 \cdots \mu_L}_{\nu_1 \nu_2 \cdots \nu_L} \\
&\qquad + 2 \sinh v (q^{\phi \sigma_1^z} X^{\pm}_{1}(\lambda - v) \, X^{\pm}_{2}(\lambda - v) \cdots X^{\pm}_{L-1}(\lambda - v) \\
&\qquad \times \Pi_R q^{\phi \sigma_1^z})^{\mu_1 \cdots \mu_{l-1} \mu_L}_{\nu_1 \cdots \nu_{L-1} \nu_L}
\end{aligned} \tag{A14}
$$

Here we have made use of the following relations

$$
\begin{aligned}
&(X^{\pm}_{L-1}(v) \, X^{\pm}_{L-2}(v) \cdots X^{\pm}_{1}(v))^{\mu_2 \cdots \mu_L \mu_1}_{\nu_1 \nu_2 \cdots \nu_L} \\
&\qquad = (\Pi_L X^{\pm}_{L-1}(v) \, X^{\pm}_{L-2}(v) \cdots X^{\pm}_{1}(v))^{\mu_1 \mu_2 \cdots \mu_L}_{\nu_1 \nu_2 \cdots \nu_L}
\end{aligned} \tag{A15}
$$

and

$$
\begin{aligned}
&-\mu_1 \nu_1 e^{\mp (\mu_1 - \nu_1) \lambda/2} q^{\phi(\mu_1 + \nu_1)} (X^{\pm}_{L-1}(v) \, X^{\pm}_{L-2}(v) \cdots X^{\pm}_{1}(v))^{\mu_2 \cdots \mu_L, -\nu_1}_{-\mu_1, \nu_2 \cdots \nu_L} \\
&\quad = (q^{\phi \sigma_1^z} X^{\pm}_{1}(\lambda - v) \, X^{\pm}_{2}(\lambda - v) \cdots X^{\pm}_{L-1}(\lambda - v) \, \Pi_R q^{\phi \sigma_1^z})^{\mu_1 \mu_2 \cdots \mu_L}_{\nu_1 \nu_2 \cdots \nu_L}
\end{aligned} \tag{A16}
$$

The relation (A15) is readily derived from the definition (3.39) of the shift operator. We can show the relation (A16) by making use of (A12) as follows.

$$
\begin{aligned}
&(X^{\pm}_{l-1}(v) \, X^{\pm}_{L-2}(v) \cdots X^{\pm}_{1}(v))^{\mu_2 \cdots \mu_L, -\nu_1}_{-\mu_1, \nu_2 \cdots \nu_L} (-\mu_1 \nu_1 e^{\mp (\mu_1 - \nu_1) \lambda/2}) \, q^{\phi(\mu_1 + \nu_1)} \\
&\quad = \sum_{\alpha_2, \ldots, \alpha_{L-1}} (-\nu_1 e^{\pm \nu_1 \lambda/2}) \, X^{\pm \mu_L, -\nu_1}_{\alpha_{L-1}, \nu_L}(v) \, X^{\pm \mu_{L-1}, \alpha_{L-1}}_{\alpha_{L-2}, \nu_{L-1}}(v) \cdots X^{\pm \mu_2, \alpha_2}_{-\mu_1, \nu_2}(v) \\
&\qquad \times (\mu_1 e^{\mp \mu_1 \lambda/2}) \, q^{\phi(\mu_1 + \nu_1)}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\alpha_2,\ldots,\alpha_{L-1}} q^{\phi(\mu_1+\nu_1)}(-\nu_1 e^{\pm\nu_1\lambda/2}) \\
&\quad (\alpha_{L-1}\nu_1 e^{\mp(\alpha_{L-1}+\nu_1)\lambda/2})\, X^{\pm\, -\alpha_{L-1},\mu_L}_{\nu_L,\nu_1}(\lambda-v) \\
&\quad (-\alpha_{L-2}\alpha_{L-1} e^{\mp(\alpha_{L-2}-\alpha_{L-1})\lambda/2}\, X^{\pm\, -\alpha_{L-2},\mu_{L-1}}_{\nu_{L-1},-\alpha_{L-1}}(\lambda-v)\cdots \\
&\qquad (-\alpha_2\alpha_3 e^{\mp(\alpha_2-\alpha_3)\lambda/2})\, X^{\pm\, -\alpha_2,\mu_3}_{\nu_3,-\alpha_3}(\lambda-v) \\
&\quad \cdot(\mu_1\alpha_2 e^{\mp(-\mu_1-\alpha_2)\lambda/2})\, X^{\pm\mu_1,\mu_2}_{\nu_2,-\alpha_2}(\lambda-v)(\mu_1 e^{\mp\mu_1\lambda/2}) \\
&= \sum_{\alpha_2,\ldots,\alpha_{L-1}} q^{\phi(\mu_1+\nu_1)}\cdot(-1)^L\cdot X^{\pm\, -\alpha_{L-1},\mu_L}_{\nu_L\nu_1}(\lambda-v)\, X^{\pm\, -\alpha_{L-2},\mu_{L-1}}_{\nu_{L-1},-\alpha_{L-1}}(\lambda-v)\cdots \\
&\qquad X^{\pm\, -\alpha_2,\mu_3}_{\nu_3,-\alpha_3}(\lambda-v)\, X^{\pm\mu_1,\mu_2}_{\nu_2,-\alpha_2}(\lambda-v) \\
&= \sum_{\alpha_2,\ldots,\alpha_{L-1}} q^{\phi(\mu_1+\nu_1)} X^{\pm\mu_1,\mu_2}_{\nu_2,-\alpha_2}(\lambda-v)\, X^{\pm\, -\alpha_2,\mu_3}_{\nu_3,-\alpha_3}(\lambda-v)\cdots \\
&\qquad X^{\pm\, -\alpha_{L-2},\mu_{L-1}}_{\nu_{L-1},-\alpha_{L-1}}(\lambda-v)\, X^{\pm\, -\alpha_{L-1},\mu_L}_{\nu_L,\nu_1}(\lambda-v) \\
&= \sum_{\beta_2,\ldots,\beta_{L-1}} q^{\phi(\mu_1+\nu_1)} X^{\pm\mu_1,\mu_2}_{\nu_2,\beta_2}(\lambda-v)\, X^{\pm\beta_2,\mu_3}_{\nu_3,\beta_3}(\lambda-v)\cdots \\
&\qquad X^{\pm\beta_{L-2},\mu_{L-1}}_{\nu_{L-1},-\beta_{L-1}}(\lambda-v)\, X^{\pm\beta_{L-1},\mu_L}_{\nu_L,\nu_1}(\lambda-v) \\
&= q^{\phi(\mu_1+\nu_1)}(X^{\pm}_1(\lambda-v)\, X^{\pm}_2(\lambda-v)\cdots X^{\pm}_{L-1}(\lambda-v))^{\mu_1\cdots\mu_{L-1}\mu_L}_{\nu_2\cdots\nu_L\nu_1} \\
&= (q^{\phi\sigma^z_1} X^{\pm}_1(\lambda-v)\, X^{\pm}_2(\lambda-v)\cdots X^{\pm}_{L-1}(\lambda-v)\, \Pi_R q^{\phi\sigma^z_1})^{\mu_1\cdots\mu_L}_{\nu_1\cdots\nu_L}
\end{aligned}
\tag{A17}
$$

where we used the fact that L is even.

In summary, we have

$$T(v;\phi) = \Pi_L X^{\pm}_{LL} + X^{\pm}_{RR}\Pi_R \tag{A18}$$

with

$$
\begin{aligned}
X^{\pm}_{LL} &= 2\sinh(\lambda-v)\, X^{\pm}_{L-1}(v)\, X^{\pm}_{L-2}(v)\cdots X^{\pm}_1(v) \\
X^{\pm}_{RR} &= 2\sinh v q^{\phi\sigma^z_1} X^{\pm}_1(\lambda-v)\, X^{\pm}_2(\lambda-v)\cdots X^{\pm}_{L-1}(\lambda-v)\, q^{\phi\sigma^z_1}
\end{aligned}
\tag{A19}
$$

A.3. The Temperley–Lieb Algebra and the Operators $S^{\pm}$, $T^{\pm}$

We shall briefly introduce matrix representations for the generators of the Temperley–Lieb algebra. Let us define operators $e^{\pm}_j$ for $j=1,\ldots, L-1$ by

$$e^{\pm}_j = \sum_{a,b,c,d=\pm 1} e^{\pm ac}_{bd} I_1 \otimes \cdots \otimes I_{j-1} \otimes E^{ab}_j \otimes E^{cd}_{j+1} \otimes I_{j+2} \otimes \cdots I_L \tag{A20}$$

where the matrix elements $e_{bd}^{\pm ac}$ are given by

$$e_{bd}^{\pm ac} = abe^{\mp(a+b)\lambda/2}\,\delta_{a+c,0}\,\delta_{b+d,0}, \qquad \text{for} \quad a, b, c, d = \pm 1 \tag{A21}$$

Utilizing the matrix representations, we can show that the operators $e_j^{\pm}$'s defined in Eq. (A20) commute with the generators $S^{\pm}$ given by (3.9) in Section 3:

$$[S^{\pm}, e_k^{+}] = 0 \qquad \text{for} \quad k = 1, 2, ..., L-1 \tag{A22}$$

and

$$[T^{\pm}, e_k^{-}] = 0 \qquad \text{for} \quad k = 1, 2, ..., L-1 \tag{A23}$$

In addition we see from (A11) that $X_j^{\pm}(v)$ can be expressed in terms of the Temperley–Lieb operators as

$$X_j^{\pm}(v) = 2\sinh(\lambda - v)\, I + 2\sinh v e_j^{\pm} \tag{A24}$$

A.4. Proof of the (Anti-)Commutation Relations in the Sector $S^z \equiv 0 (\mathrm{mod}\, N)$

In the sector $S^z \equiv 0(\mathrm{mod}\, N)$ we need only consider untwisted operators with $\phi = 0$. The product of the operators $X_j^{\pm}(v)$'s can be written in terms of the Temperley–Lieb generators $e_j^{\pm}$'s

$$\begin{aligned} &X_{L-1}^{\pm}(v)\, X_{L-2}^{\pm}(v) \cdots X_1^{\pm}(v) \\ &\quad = (\rho(v)\, I + f(v)\, e_{L-1}^{\pm})(\rho(v)\, I + f(v)\, e_{L-2}^{\pm}) \cdots (\rho(v)\, I + f(v)\, e_1^{\pm}) \end{aligned} \tag{A25}$$

where $\rho(v) = 2\sinh(v - \lambda)$ and $f(v) = 2\sinh v$. Thus, using (A22) we have for all q the commutation relation

$$[S^{\pm}, X_{L-1}^{+}(v)\, X_{L-2}^{+}(v) \cdots X_1^{+}(v)] = 0 \tag{A26}$$

and

$$[S^{\pm}, X_1^{+}(\lambda - v)\, X_2^{+}(\lambda - v) \cdots X_{L-1}^{+}(\lambda - v)] = 0 \tag{A27}$$

and for the operators $T^{\pm}$ we use (A23) to obtain

$$[T^{\pm}, X_{L-1}^{-}(v)\, X_{L-2}^{-}(v) \cdots X_1^{-}(v)] = 0 \tag{A28}$$

$$[T^{\pm}, X_1^{-}(\lambda - v)\, X_2^{-}(\lambda - v) \cdots X_{L-1}^{-}(\lambda - v)] = 0 \tag{A29}$$

Then using (3.38), (A18) and (A26)–(A29) we find for $S^z \equiv 0 (\mathrm{mod}\ N)$ the (anti)commutation relations with the transfer matrix

$$S^{\pm(N)} T(v) = q^N T(v)\, S^{\pm(N)} \tag{A30}$$

$$T^{\pm(N)} T(v) = q^N T(v)\, T^{\pm(N)} \tag{A31}$$

or equivalently

$$[S^{\pm(N)}, \Pi_L T(v)] = 0 \tag{A32}$$

$$[T^{\pm(N)}, \Pi_L T(v)] = 0 \tag{A33}$$

A.5. The Commutation Relations in the Sector $S^z \equiv n (\mathrm{mod}\ N)$

We conclude this appendix by considering the commutation relation of $(S^+)^n (T^-)^n$ with the transfer matrix $T(v)$. From (A18) and (A26) we have

$$\begin{aligned}(\Pi_R T(v;0))(S^+)^n\,(T^-)^n &= (X^+_{LL} + \Pi_R X^+_{RR} \Pi_R)(S^+)^n \cdot (T^-)^n \\ &= ((S^+)^n\, X^+_{LL} + \Pi_R X^+_{RR} \Pi_R (S^+)^n) \cdot (T^-)^n\end{aligned} \tag{A34}$$

We further study the second term in this expression by using (3.66) twice to obtain for arbitrary q in the sector $S^z = n > 0$

$$\begin{aligned}&\Pi_R X^+_{RR} (\Pi_R (S^+)^n)(T^-)^n \\ &\quad = \Pi_R X^+_{RR} \{(S^+)^n + q^{n-1}[n](S^+)^{n-1}\, S^+_L (q^{-2S^z} - 1)\}\; q^{n\sigma^z_L} \Pi_R (T^-)^n \\ &\quad = \Pi_R X^+_{RR} (S^+)^n\; q^{n\sigma^z_L} \Pi_R (T^-)^n \\ &\quad = (\Pi_R (S^+)^n)\; X^+_{RR} \Pi_R Q^{n\sigma^z_1} (T^-)^n \\ &\quad = \{(S^+)^n + q^{n-1}[n](S^+)^{n-1}\, S^+_L (q^{-2S^z} - 1)\} \\ &\qquad \times q^{n\sigma^z_L} \Pi_R X^+_{RR} \Pi_R q^{n\sigma^z_1} (T^-)^n \\ &\quad = (S^+)^n\, \Pi_R q^{n\sigma^z_1} X^+_{RR} \Pi_R q^{n\sigma^z_1} (T^-)^n\end{aligned} \tag{A35}$$

which when used in (A34) yields

$$\begin{aligned}(\Pi_R T(v;0))(S^+)^n\,(T^-)^n\,|n\rangle &= ((S^+)^n\, X^+_{LL} + \Pi_R X^+_{RR} \Pi_R (S^+)^n)(T^-)^n \\ &= (S^+)^n\,(X^+_{LL} + \Pi_R q^{n\sigma^z_1} X^+_{RR} \Pi_R q^{n\sigma^z_1})(T^-)^n \\ &= (S^+)^n\, \Pi_R T(v;n)(T^-)^n\end{aligned} \tag{A36}$$

Similarly we find

$$\Pi_R T(v; n)(T^-)^n = (T^-)^n \, \Pi_R T(v; 0) \tag{A37}$$

Thus, we obtain the commutation relation valid for $S^z = n > 0$ and all q

$$[T(v), (S^+)^n \, (T^-)^n] = 0 \tag{A38}$$

When $q^{2N} = 1$ this argument leading to (A38) immediately extends to $S^z \equiv n(\mathrm{mod}\ N)$.

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